CONTINUITY

ELEVEN SKETCHES
FROM THE PAST OF MATHEMATICS

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JERZY MIODUSZEWSKI

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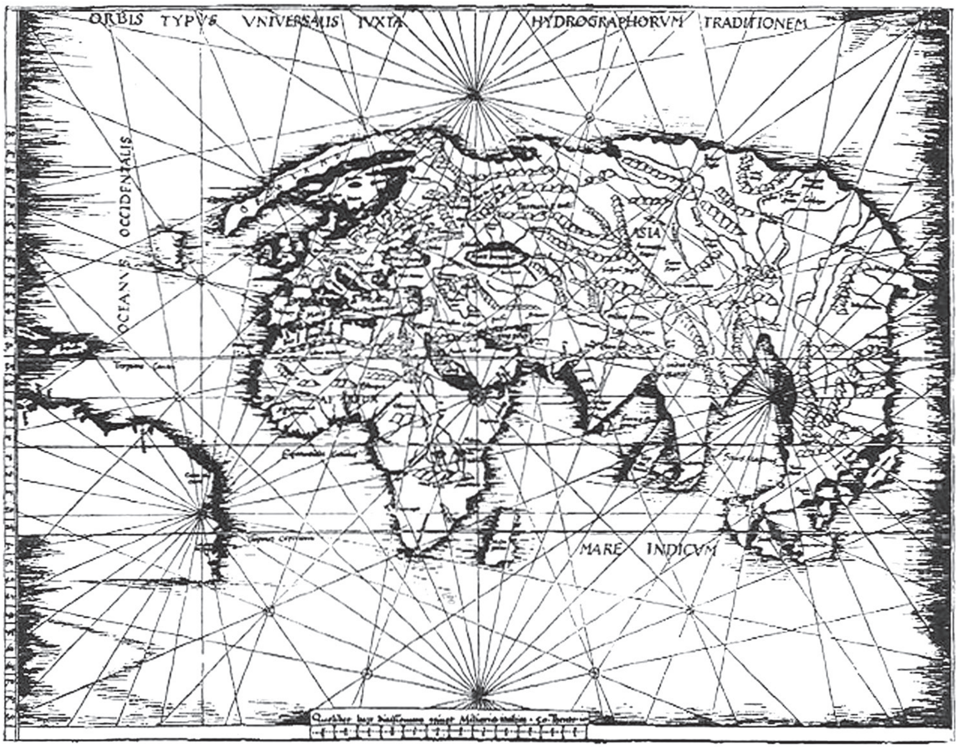

TRANSLATED 2008–2015
BY PROFESSOR ABE SHENITZER

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From the author

The book was written in the eighties of the last century. Being encouraged by the editorial board of monthly *Delta* in the person of Professor Marek Kordos, the author's first aim was a collection of essays about Peano maps, lakes of Wada, and several singularities of real functions. But it was the time when university duties stopped and the author could freely meditate whether this curious mathematics had its roots in the forgotten past. He remembered old authors who began their books with the words "already the ancient Greeks…."

The celebrated nineteenth century, the century of concepts, was preceded by the century of calculations. Going further back we can see Newton, but what and who was there before? Were the centuries between the Ancients and Newton a vacuum in mathematical sciences? Accidentally, the treatise *De continuo* by Thomas Bradwardine, the Archbishop of Canterbury, led the author into an unknown and strange world of medieval scholastic thought, showing to him the lost thread joining our times with Zeno, Aristotle and Democritus.

However, to find this forgotten link a step should be taken beyond pure mathematical thinking. In this extended surrounding we can observe the unity of mathematical concepts being non-existent in the realm of pure mathematics.

The translation into English is a gift from Professor Abe Shenitzer. Although the translation runs as closely as possible to the Polish original text, the author has a right to regard the book as a joint work with the Translator. The first chapter of the original text, according to Translator's suggestion, was "somewhat baffling for reading." That is why the author decided not to enclose it in the English version and consequently the numeration of chapters is shifted by one with respect to the Polish edition.

The author is truly indebted to his daughter, Elżbieta and granddaughter, Berenika for scanning the drawings and consolidating the text consisting of separate mails into a computer whole and forming it into a volume for the home use, which was a base for the further work.

The drawings are taken from the Polish edition; those of pure mathematical character were created professionally by the late Krzysztof Biesaga.

The publication was possible thanks to the goodwill and the support of the Faculty of Mathematics, Physics and Chemistry in the person of the Dean Professor Alicja Ratuszna, and the kind cooperation with the University of Silesia Press. The author expresses his special gratitude to MSc Joanna Zwierzyńska for her careful look at the final version of the text and making it more coherent, and for saving the text from numerous inaccuracies.

Introduction

There are two areas of mathematics, namely, arithmetic and geometry. They are independent, yet clearly separated. Arithmetic deals with numbers, geometry deals with space. Whereas the notion of number is rooted in our thinking that most creators of mathematics were inclined to accept it without discussion, views on space have always been subject to deep splits. Whether space should be treated as a mathematical object — that is as an object of thought — or as a physical object is a question which we will not answer. Parmenides, one of the first philosophers of nature whose views we will have occasion to investigate, identified space with ideal existence, and thus with existence that is invariant, homogeneous, infinite, and forming an entity.

The people noted more specific characteristic of space. One of them is *continuity.*

This characteristic of space is so much part of our notions that we lose our way in its analysis. In ancient Greece the continuity of space meant like the possibility of subdividing it indefinitely. This was the view of Anaxagoras who said that "there is no least in the small." Translated into non-archaic language, this means that one can subdivide every part of space. Aristotle took this characteristic of space as the starting point of his investigations. But there is another characteristic of continuity which ensure the cohesiveness of continuous existence: two parts into which we separate it mentally adhere to one other. A mathematical formulation of this characteristic was discovered only a little more than a hundred years ago.

A continuous object, that is, one infinitely divisible and cohesive, has been called already in antiquity a *continuum*. The root of this word is the Latin *continere*, whose Greek prototype is *syn-echein*, which roughly means to *bond*.

Space is not the only object to which we ascribe continuous structure. The intensity of stream, or of color, seem to have this quality. But, above all, it is the flow of time that is continuous.

A loose and free structure, composed of isolated elements, is the opposite of a continuous structure. Such a structure is said to be *discrete*. The word "discrete" is derived from the Latin *discretus*, separate, detached from other things. "Discrete" thus means "consisting of, or pertaining to, distinct and individual parts."

The numbers

$$1, 2, 3, \ldots$$

form a discrete structure.

Could space be discrete? This cannot be ruled out a priori. Nor can we rule out of possibility that the flow of time might be discrete.

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Geometry, the mathematical science of space, has also another, more mundane origin. The two relevant Geek word are *gea* — land (we mean arable land) and *metrein* — to measure. Proclus (ca. 410—485), a commentator of works of his predecessors, wrote that "Many people assert that geometry was invented by Egyptians for measurement of land. They needed it because the inhabitants of the Nile washed out balks."

From balks to infinitely divisible existence — a breathtaking span.

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Space is a composite object made up of elements that enable us to realize the nature of the whole. We single out *points* — places in space. This is not a definition but just another term of language. Points are not parts of space: we do not attribute them a material nature even when we are prepared to attribute a material nature to space. They are not a raw material out of which space, or a part of it, is composed. When we think of a point, we think of its location. A point is a synonym of its neighbourhood. Only if the space is not uniform, these neighbourhoods may be different.

Nevertheless, we are willing to imagine points as independent existences, and the thought that they could be the raw material of space does not always strike us as alien. This dilemma is one of the difficulties we encounter when we think of the notion of a continuum.

Another difficulty is the infinitude of space, a notion which suggest itself irresistibly when we think of *straight lines*, yet another element of space.

After a few attempts we give up the idea of defining a straight line. It seems to be as primitive as the concept of space. One can also adopt the reverse view point: it is straight lines that suggest to us the notion of space. We see and move along straight lines. Moving along the straight line, we move towards an objective. We are not always sure of the possibility of reaching it. Hence straight lines give us the initial sense of the possible nature of the infinite.

Planes are yet another element. We see in space at least one plane, the plane we seem to be in. The initial stage of geometry codifies our notions related to our staying in that plane. Space notions came later. Then we begin to notice other planes as well.

The mutual disposition of points, straight lines and planes is subject to definite rules (such as say, that two different straight lines can have at most one common point, that they adhere to planes, and so on). That are truths that must be accepted without proof (which does not mean on faith). Such truths are called *postulates*. It is arguable whether postulates are facts so obvious that nature thrusts them before our eyes and all we need do is note them, or whether they statements are the result of slowly growing knowledge that is finally spelled out, knowledge of which we do not know whether it is final and beyond doubt. The evolution of geometry tells us that what is true is the latter rather than the former.

It is also arguable whether the formation of geometric postulates belongs in the domain of mathematics, or philosophy, the guide of learning. Aristotle was believed that the issue belongs to philosophy. This statement should be interpreted as saying that the issue is metamathematical, i.e. lies beyond mathematics.

We attribute the quality of continuity to plans and straight lines.

But straight lines are continua with the earnest structure. A point divides straight line into two parts, each of which is again a continuum. This property of a straight line enables us to order the set of its points. We say that a straight line is an *ordered* continuum. We also say that it is one-dimensional. Neither a plane nor space have this property.

What is space? Why does it exceed our imagination and why must a child learn about it? Why do even accomplished painters lose their way when dealing with perspective, a subject whose knowledge is only a few century old, and produce either "flat" paintings or "space" paintings that are frequently flawed? Why can't we exit from space into an extra dimension the way we exit from a plane? Is it because of a limitation of our senses or is it because of the nature of space? While the first of these views is very popular and opens the door to a variety of speculations, the three dimensionality of space is a physical fact; no mathematical premise supports the number 3. Kant linked the number of dimensions with the form of the law of gravitation. Can it be that counting dimensions is a necessity of our thought processes?

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Time is very troublesome. The 19th century provided a simple mathematical description of time but behind it hides a physical phenomenon that is hard to grasp. There is also a subjective sense of time. The two are connected. Explanation of this connection is a task of natural sciences: for physics, physiology, and psychology. In spite of its vagueness, time is subjectively the most continuous of all continuities: if we cannot imagine a break in the space then we cannot possibly imagine a break in time.

It seems that time is a stream of events with a direction. It isn't clear whether the notion of direction of time is due to our senses or is part of the nature of things. Time seems to flow continuously. If not much is going on, then we notice changes of the intensity of its flow, momentary atrophies and turnings. We seem to flow with the stream. We do not know if the flow of time is everywhere the same and whether it will always be the same. We cannot imagine its ever coming to an end and its ever beginning. We experience the physical nature of time most having intensely when we can turn time back. Preconditions for this are: a small number of phenomena and not much happening. Then we can turn the time back by restoring earlier positions of moved objects. To turn the time back in the full sense of the word we would need all the energy in the world, if not more. Aristotle, with Plato in mind, said that "Some claims that time is the motion of the whole world." St. Augustine agreed with Plato and thought that time began at the moment of creation, and added that before that moment eternity ruled.

We tend to think of a moment as a point separating the past of the future. This means that we are willing to treat time as the ordered continuum, a universal continuum for all phenomena, but, strictly speaking, we never ascertain this universality. Each range of phenomena seems to have its own time stream. The time notion we use is always a strand we attribute to the stream of phenomena in which we move. In that strand a moment seems to have a definite content. In mathematical problems we restrict phenomena so that time takes on the structure of a straight line.

The ancients removed time from the range of mathematics. Their geometry — as Aristotle stated succintly — was limited to consideration of *motionless existences*. They had definite reasons for so doing. We will talk of these reasons. Modern mathematics has included time in its deliberation as a schematic existence devoid of all the varied properties suggested by its nature.

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We speak of space and time as of things. We have no right to do this because these are qualities of things rather than things, qualities we might call spaceness and variability. But when speaking about qualities of things we sometimes find it convenient to elevate them to the level of things. Then we forget about the origin of the new existences and treat them like things. Plato called these existences *ideas*, and maintained that they are the only things worthy of deliberation. Let's not argue about this. An issue more worthy of argument is probably the issue of the origin of ideas. In spite of the fact that we are their makers (or, at least, we think we are), we make them as a result of the pressure of phenomena, and this endows them with a quality of objectivity. If we do not want to limit ourselves to the manipulation of objects and events, then ideas are indispensable for our thinking. We fix their properties so as to enable to think about these properties as if they were characteristics of external objects.

But it is an exaggeration to follow the believers in Plato for whom the world of objects and phenomena is a mere reflection of the world of ideas. We can go further in this opposition to Plato, like Aristotle we can say that ideas are the only things we can investigate in a rigorous manner.

In spite of the fact that ideas evolve, the evolution of mathematical ideas is very slow. This gives the impression that the structure of mathematical knowledge grows like a building. The notion of number does not change, and when we look at the three millennia of the evolution of geometry, to the period for which we have documentary evidence, the changes of concepts are minimal. The concepts of physics are less durable. But we hasten to add that it took two millennia to replace the physics of Aristotle with its opposite, the physics of Newton. Some claim that the most durable principles are the principles of logic.

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Time to pose a more basic question. To what extent are the mathematical notions we form independent of the way we observe or even of the nature of our senses?

This question was posed by Kant. Roughly speaking his answer was that in our choice of motions bearing on time and space we are limited by our nature. Once equipped with such notions — whether inherited, learned at an early age, or picked up with the rest of the culture of our environment — we use them in fixed form.

According to extreme views connected with this orientation, man is equipped with a sense of time and space which imposes a definite pattern on the knowledge he forms. We cannot completely reject this possibility, but in line with what we've said thus far, we state a reservation. Even if it is true that our sense of time and space depends on the limitations of our nature, this sense was shaped under the influence of the outside world, and thus contains a general cognitive element. To use Kant's terminology, this is *a cognitive element a priori*.

Kant's views are a good reference to a veritable maze of presentations philosophy which can serve and that admits to mathematics. We took a step away from Kant's view in a direction that admits the evolution of what Kant called reason. But one can take a step in a direction that ascribes to reason in Kant's sense invariability and absolute infallibility. The invariability of mathematical truths seems to justify this view. Many thinkers, such as the Pythagoreans, Parmenides, Plato (the key representative of this viewpoint), St. Augustine, and among more recent figures, Bolzano and Cantor, are inclined to accept it.

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For the Greeks, the notion of continuum emerged from the philosophy of nature, that is, contemporary physics. Attempts of its mathematization failed. The famous aporia of Zeno of Elea paralyzed these attempts. Such failed attempts are found in the works of Aristotle, which include an accent of his own view. Aristotle concentrated the key difficulty in the question whether the continuum can be viewed as made up of points.

An affirmative answer leads to difficulties. Aristotle was sufficiently open minded to admit that it also leads to a logical contradiction. But the negative answer deprives us of methods.

Attempts were made to get around these difficulties by erecting certain thought barriers. The construction of Euclid's *Elements* rules out the possibility of stating Zeno's aporias in the language used there.

We know more of the continuum than the Greeks, but the area of ignorance has not decreased. Every now and again discoveries are made. They are undoubtedly important but are unnecessarily advertised as epochal, discoveries that claim to have solved the problem.

We will try to show that this view is false. We will give a historical account of the problem and show how philosophers and mathematicians, both famous and not very famous, lost they way in the labyrinth of the continuum, what was the outcome of their efforts, and in what sense their labors, so seemingly Sisyphian, were actually not.

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